Noncommutative Kaluza-Klein Theory

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Abstract

Efforts have been made recently to reformulate traditional Kaluza-Klein theory by using a generalized definition of a higher-dimensional extended space-time. Both electromagnetism and gravity have been studied in this context. We review some of the models which have been proposed, with a special effort to keep the mathematical formalism to a very minimum.

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1 Introduction and Motivation

The simplest definition of noncommutative geometry is that it is a geometry in which the coordinates do not commute. Perhaps not the simplest but certainly the most familiar example is the quantized version of a 2-dimensional phase space described by the 'coordinates' p and q. This example has the advantage of illustrating what is for us the essential interest of noncommutative geometry as expressed in the Heisenberg uncertainty relations: the lack of a well-defined notion of a point. 'Noncommutative geometry is pointless geometry.' The notion of a point in space-time is often an unfortunate one. It is the possibility in principle of being able to localize a particle at any length scale which introduces the ultraviolet divergences of quantum field theory. It would be interesting then to be able to modify the coordinates of space and time so that they become noncommuting operators. By analogy with quantum mechanics one could then expect points to be replaced by elementary cells. This cellular structure would serve as an ultraviolet cut-off similar to a lattice structure. The essential difference is that it is possible in principle to introduce the cellular structure without breaking Lorentz invariance.

When a physicist calculates a Feynman diagram he is forced to place a cut-off Λ on the momentum variables in the integrands. This means that he renounces any interest in regions of space-time of dimension less than Λ^{-1}. As Λ becomes larger and larger the forbidden region becomes smaller and smaller. The basic assumption which we make is that this forbidden region cannot, not only in practice but even in principle, become arbitrarily small. There is a fundamental length scale below which the notion of a point makes no sense. The simplest and most elegant, if not the only, way of doing this in a Lorentz-invariant way is through the introduction of non-commuting coordinates, exactly as in quantum mechanics.

To illustrate in more detail the analogy with quantum mechanics it is of interest to examine the phase space of a classical particle moving in a plane. In the language of quantum mechanics it is described by two position operators (q^1, q^2) and two momentum operators (p_1, p_2). These four operators all commute; they can be simultaneously measured and the eigenvalues can be considered as the coordinates of the points of a 4-dimensional space. When the system is quantized they no longer commute; they satisfy the canonical commutation relations

$$[q^1, p_1] = i\hbar, \qquad [q^2, p_2] = i\hbar. \tag{1}$$

Because of this there is no longer a notion of a point in phase space since one cannot measure simultaneously the position and momentum of a particle to arbitrary precision. However phase space can be thought of as divided into cells of volume $(2\pi\hbar)^2$ and it is this cellular structure which replaces the point structure. If the classical phase space is of finite total volume there are a finite number of cells and the quantum system has a finite number of possible states. A function on phase space is defined then by a finite number of values.

In the presence of a magnetic field H normal to the plane the momentum operators must be further modified. They are replaced by the minimally-coupled expressions and they also cease to commute:

$$[p_1, p_2] = ieH\hbar. \tag{2}$$

This introduces a cellular structure in the momentum plane. It becomes divided into 'Landau cells' of area proportional to $eH\hbar$. Consider in this case the divergent integral

$$I = \int \frac{d^2p}{p^2}. \tag{3}$$

The commutation relations (2) do not permit p_1 and p_2 simultaneously to take the eigenvalue zero and the operator p^2 is bounded below by $eH\hbar$. The magnetic field acts as an infrared cut-off. If one adds an *ad hoc* ultraviolet cut-off Λ then p^2 is bounded also from above and the integral becomes finite:

$$I \sim \log(\frac{\Lambda^2}{eH\hbar}). \tag{4}$$

If the position space were curved, with constant Gaussian curvature K, one would have (2) with the minimally-coupled expressions for the momentum and with $eH\hbar$ replaced by $K\hbar^2$. One would obtain again an infrared regularization for I.

One can also suppose the coordinates of position space to be replaced by two operators which do not commute:

$$[q^1, q^2] = i\bar{k}. \tag{5}$$

The constant $\bar{k}$ determines a new length scale which has no *a priori* relation with $\hbar$ any more than (2) has a relation with (1). By the new uncertainty relation there is no longer a notion of a point in position space since one cannot measure both coordinates simultaneously but as before, position space can be thought of as divided into cells. If we consider for example the divergent integral I and use the same logic that led to (4) we find that the commutation relations (5) introduce an ultraviolet cut-off. If we introduce also a constant Gaussian curvature and use the equivalent of (2) we have

$$I \sim \log(\bar{k}K). \tag{6}$$

The integral has become completely regularized.

Although the ultimate ambition of noncommutative geometry (in physics) is is to introduce a noncommutative version of space-time and to use it to describe quantum gravity, we shall here address the much more modest task of considering a modified version of Kaluza-Klein theory in which the hidden 'internal' space alone is described by a noncommutative geometry. The rational for this is the fact that the hidden dimensions, if at all, are small. In the following Section we briefly recall the basics of the standard version of Kaluza-Klein theory but in a notation which makes it natural to pass to the modified version. This means above all that we must introduce the notion of a differential in a rather abstract way since later we shall be forced to take differentials of matrices and other objects which in the usual sense of the word do not possess derivatives. This is the only technical part of the article from a mathematical point of view. In Section 3 we give a very rudimentary introduction to some of the more elementary aspects of noncommutative geometry, just sufficient so as to be able to pass in the following Section 4 to the description of the modified versions of Kaluza-Klein theory. This is the central section. In it we describe models based on electromagnetism which purport to describe various aspects of the standard model of electro-weak and strong interactions. Only in the concluding Section 5 shall we be interested in the gravitational field. We shall there describe what we consider might be a relation between noncommutative geometry and classical and/or quantum gravity.

The subject has evolved considerably since a similar review was written in 1988 [71]. For an sampling of the early history of ideas on the micro-texture of space-time we refer to Section 1.3 of the book by Prugovečki [88] as well as to the review articles by Kragh & Carazza [69] and Gibbs [47].

2 Kaluza-Klein Theory

The question of whether or not space-time has really 4 dimensions, and why, has been debated for many years. One of the first negative answers was given by Kaluza [64] and Klein [68] in their attempt to introduce extra dimensions in order to unify the gravitational field with electromagnetism. Einstein & Bergmann [44] suggested that at sufficiently small scales what appears as a point will in fact be seen as a circle. Later, with the advent of more elaborate gauge fields, it was proposed that this internal space could be taken as a compact Lie group or something more general. The great disadvantage of these extra dimensions is that they introduce even more divergences in the quantum theory and lead to an infinite spectrum of new particles. In fact the structure is strongly redundant and most of it has to be discarded. An associated problem is that of localization. We cannot, and indeed do not wish to have to, address the question of the exact position of a particle in the extra dimensions any more than we wish to localize it too exactly in ordinary space-time. We shall take this as motivation for introducing in Section 4 a modification of Kaluza-Klein theory with an internal structure which is described by a noncommutative geometry and in which the notion of a point does not exist. As particular examples of such a geometry we shall choose only internal structures which give rise to a finite spectrum of particles.

In its local aspects Kaluza-Klein theory is described by an extended space-time of dimension $N = 4 + k$ with coordinates $x^i = (x^\alpha, x^a)$. The x^α are the coordinates of space-time which, except for the last section, one can consider to be Minkowski space; the x^a are the coordinates of the internal space, which in this section will be implicitly supposed to be space-like and 'small'. In Section 4 it will be of purely algebraic nature and not necessarily 'small'.

One of the most important tools in differential geometry is the differential of a function and the most important advance in noncommutative geometry has been the realization by Connes [17, 19] that the differential has a natural extension to the non-commutative case. We shall define a differential by a set of simple rules which makes it obvious that it is equivalent to a derivative and ask the reader to believe that the rules have a rigorous and natural mathematical foundation. He will see that they are quite easy to manipulate in the simple noncommutative geometries we consider in Section 3

A 1-form is a covariant vector field A_i, which we shall write as $A = A_i dx^i$ using a set of basis elements dx^i. A 2-form is an antisymmetric 2-index covariant tensor F_{ij} which we shall write as

$$F = \frac{1}{2} F_{ij} dx^i dx^j \tag{7}$$

using the product of the basis elements. This product is antisymmetric

$$dx^i dx^j = -dx^j dx^i \tag{8}$$

but otherwise has no relations. Higher-order forms can be defined as arbitrary linear combination of products of 1-forms. A p-form can be thus written (locally) as

$$\alpha = \frac{1}{p!} \alpha_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_p}. \tag{9}$$

The coefficients $\alpha_{i_1 \cdots i_p}$ are smooth functions and completely antisymmetric in the p indices.

Let $\mathcal{A}$ be the set of complex-valued functions on the extended space-time. Since the product of two functions can be defined, and is independent of the order, $\mathcal{A}$ is a commutative algebra. We define $\Omega^0(\mathcal{A}) = \mathcal{A}$ and for each p we write the vector space

of p-forms as $\Omega^p(\mathcal{A})$. Each $\Omega^p(\mathcal{A})$ depends obviously on the algebra $\mathcal{A}$ and, what is also obvious and very important, it can be multiplied both from the left and the right by the elements of $\mathcal{A}$. It is easy to see that $\Omega^p(\mathcal{A}) = 0$ for all $p \geq n+1$. We define $\Omega^*(\mathcal{A})$ to be the set of all $\Omega^p(\mathcal{A})$. We have seen that $\Omega^*(\mathcal{A})$ has a product given by (8). It is a graded commutative algebra. It can be written as a sum

$$\Omega^*(\mathcal{A}) = \Omega^+(\mathcal{A}) \oplus \Omega^-(\mathcal{A}) \tag{10}$$

of even forms and odd forms. The A is an odd form and F is even. The algebra $\mathcal{A}$ is a subalgebra of $\Omega^+(\mathcal{A})$.

Let f be a function, an element of the algebra $\mathcal{A} = \Omega^0(\mathcal{A})$. We define a map d from $\Omega^p(\mathcal{A})$ into $\Omega^{p+1}(\mathcal{A})$ by the rules

$$df = \partial_i f \, dx^i, \qquad d^2 = 0. \tag{11}$$

It takes odd (even) forms into even (odd) ones. From the rules we find that

$$dA = d(A_i dx^i) = \frac{1}{2}(\partial_i A_j - \partial_j A_i)dx^i dx^j = F \tag{12}$$

if we set

$$F_{ij} = \partial_i A_j - \partial_j A_i. \tag{13}$$

From the second rule we have

$$dF = 0. \tag{14}$$

It is easy to see that if α is a p-form and β is a q-form then

$$\alpha\beta = (-1)^{pq}\beta\alpha, \qquad d(\alpha\beta) = (d\alpha)\beta + (-1)^p \alpha d\beta. \tag{15}$$

The couple $(\Omega^*(\mathcal{A}), d)$ is called a differential algebra or a differential calculus over $\mathcal{A}$. We shall see later that $\mathcal{A}$ need not be commutative and $\Omega^*(\mathcal{A})$ need not be graded commutative. Over each algebra $\mathcal{A}$, be it commutative or not, there can exist a multitude of differential calculi. This fact makes the noncommutative version of Kaluza-Klein theory richer than the commutative version. As a simple example we define what is known as the universal calculus $(\Omega_u^*(\mathcal{A}), d_u)$ over the commutative algebra of functions $\mathcal{A}$. We set, as always, $\Omega_u^0(\mathcal{A}) = \mathcal{A}$ and for each $p \geq 1$ we define $\Omega_u^p(\mathcal{A})$ to be the set of p-point functions which vanish when any two points coincide. It is obvious that $\Omega_u^p(\mathcal{A}) \neq 0$ for all p, whatever N. There is a map d from $\Omega_u^p(\mathcal{A})$ into $\Omega_u^{p+1}(\mathcal{A})$ given in the lowest order by

$$(d_u f)(x^i, y^i) = f(y^i) - f(x^i). \tag{16}$$

In higher orders it is given by a similar sort of alternating sum defined so that $d_u^2 = 0$. The algebra $\Omega_u^*(\mathcal{A})$ is not graded commutative. It is defined for arbitrary functions, not necessarily smooth, and it has a straightforward generalization for arbitrary algebras, not necessarily commutative.

What distinguishes the usual differential calculus is the fact that it is based on derivations. The derivative $\partial_i f$ of a smooth function f is a smooth function. We use the word derivation to distinguish the map ∂_i from the result of the map $\partial_i f$. A general derivation is a linear map X from the algebra into itself which satisfies the Leibniz rule: $X(fg) = (Xf)g + f(Xg)$. In the case we are presently considering a derivation can always be written (locally) in terms of the basis ∂_i as $X = X^i \partial_i$. Such is not always the case. The relation between d and ∂_i is given by

$$df(\partial_i) = \partial_i f. \tag{17}$$

This equation has the same content as the first of (11). One passes from one to the other by using the particular case

$$dx^i(\partial_j) = \delta^i_j. \tag{18}$$

The basis dx^i is said to be dual to the basis ∂_i. The derivations form a vector space (the tangent space) at each point, and (17) defines df as an element of the dual vector space (the cotangent space) at the same point. In the examples we consider in Section 3 there are no points but the vector spaces of derivations are still ordinary finite-dimensional vector spaces. Over an arbitrary algebra which has derivations one can always define in exactly the same manner a differential calculus based on derivations. These algebras have thus at least two, quite different, differential calculi, the universal one and the one based on the set of all derivations.

To form tensors one must be able to define tensor products, for example the tensor product $\Omega^1(\mathcal{A}) \otimes_\mathcal{A} \Omega^1(\mathcal{A})$ of $\Omega^1(\mathcal{A})$ with itself. We have here written in subscript the algebra $\mathcal{A}$. This piece of notation indicates the fact that we identify $\xi f \otimes \eta$ with $\xi \otimes f\eta$ for every element f of the algebra, a technical detail which is important in the applications of Section 4. It means also that one must be able to multiply the elements of $\Omega^1(\mathcal{A})$ on the left and on the right by the elements of the algebra $\mathcal{A}$. If $\mathcal{A}$ is commutative of course these two operations are equivalent. When $\mathcal{A}$ is an algebra of functions this left/right linearity is equivalent to the property of locality. It means that the product of a function with a 1-form at a point is again a 1-form at the same point, a property which distinguishes the ordinary product from other, non-local, products such as the convolution. In the noncommutative case there are no points and locality can not be defined; it is replaced by the property of left and/or right linearity with respect to the algebra.

To define a metric and covariant derivatives on the extended space-time we set

$$\theta^i = dx^i \tag{19}$$

in the absence of a gravitational field. We have then

$$d\theta^i = 0. \tag{20}$$

The extended Minkowski metric can be defined as the map

$$g(\theta^i \otimes \theta^j) = g^{ij} \tag{21}$$

which associates to each element $\theta^i \otimes \theta^j$ of the tensor product $\Omega^1(\mathcal{A}) \otimes_\mathcal{A} \Omega^1(\mathcal{A})$ the contravariant components g^{ij} of the (extended) Minkowski metric. There are of course several other definitions of a metric which are equivalent in the case of ordinary geometry but the one we have given has the advantage of an easy extension to the noncommutative case. The map g must be bilinear so that we can define for arbitrary 1-forms $\xi = \xi_i \theta^i$ and $\eta = \eta_i \theta^i$

$$g(\xi \otimes \eta) = \xi_i \eta_j g(\theta^i \otimes \theta^j) = \xi_i \eta_j g^{ij}. \tag{22}$$

We introduce a gauge potential by first defining a covariant derivative. Let ψ be a complex-valued function which we shall consider as a 'spinor field' with no Dirac structure and let $\mathcal{H}$ be the space of such 'spinor fields'. A covariant derivative is a rule which associates to each such ψ in $\mathcal{H}$ a spinor-1-form $D\psi$. It is a map

$$\mathcal{H} \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_\mathcal{A} \mathcal{H} \tag{23}$$

from $\mathcal{H}$ into the tensor product $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}$. In the absence of any topological complications the function $\psi = 1$ is a spinor field and we can define a covariant derivative by the rule

$$D1 = A \otimes 1. \tag{24}$$

The (local) gauge transformations are the complex-valued functions with unit norm and so A must be a 1-form with values in the Lie algebra of the unitary group U_1, that is, the imaginary numbers. An arbitrary spinor field ψ can always be written in the form $\psi = f \cdot 1 = 1 \cdot f$ where f is an element of the algebra $\mathcal{A}$. The extension to ψ of the covariant derivative is given by the Leibniz rule:

$$D\psi = df \otimes 1 + A \otimes f = d\psi \otimes 1 + A \otimes \psi, \tag{25}$$

an equation which we shall simply write in the familiar form $D\psi = d\psi + A\psi$. Using the graded Leibniz rule

$$D(\alpha\psi) = d\alpha \otimes \psi + (-1)^p \alpha D\psi, \tag{26}$$

the covariant derivative can be extended to higher-order forms and the field strength F defined by the equation

$$D^2\psi = F\psi. \tag{27}$$

To introduce the gravitational field it is always possible to maintain (21) but at the cost of abandoning (20). This is known as the moving-frame formalism. In the presence of gravity the dx^i become arbitrary 1-forms θ^i. The differential df can still be written

$$df = e_i f \theta^i \tag{28}$$

in the form (11) provided one introduces modified derivations e_i. We shall give explicitly expressions for such derivations in a noncommutative example in Section 3. An equation

$$df(e_i) = e_i f \tag{29}$$

equivalent to (17) can be written if one imposes the relations

$$\theta^i(e_j) = \delta^i_j. \tag{30}$$

The θ^i are a (local) basis of the 1-forms dual to the derivations e_i exactly as the dx^i are dual to the ∂_i. Equation (20) must be replaced by the structure equations

$$d\theta^i = -\frac{1}{2}C^i{}_{jk}\theta^j\theta^k \tag{31}$$

which express simply the fact that the differential of a 1-form is a 2-form and can be thus written out in terms of the (local) basis $\theta^i\theta^j$. The structure equations can normally not be written globally and in the noncommutative case such equations do not in general make sense because the differential forms need not have a basis.

A covariant derivative is a rule which associates to each covariant vector ξ a 2-index covariant tensor $D\xi$. It is a map

$$\Omega^1(\mathcal{A}) \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \tag{32}$$

from $\Omega^1(\mathcal{A})$ into the tensor product $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$. On the extended space-time we can define a covariant derivative by the rule

$$D\theta^i = -\Gamma^i{}_{jk}\theta^j \otimes \theta^k. \tag{33}$$

The extension to arbitrary $\xi = \xi_i \theta^i$ is given by the Leibniz rule:

$$D\xi = d\xi_i \otimes \theta^i - \xi_i \Gamma^i{}_{jk} \theta^j \otimes \theta^k. \tag{34}$$

Using again a graded Leibniz rule, D can be extended to higher-order forms and the curvature Ω defined by the equation

$$D^2\xi = -\Omega\xi = -\xi_i \Omega^i{}_j \otimes \theta^j. \tag{35}$$

The curvature is the field strength of the gravitational field. The minus sign is an historical convention. One can be write $\Omega^i{}_j$ in the form

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl} \theta^k \theta^l \tag{36}$$

an equation which defines the components $R^i{}_{jkl}$ of the Riemann tensor.

The (local) gauge transformations are the functions with values in the (local extended) Lorentz group. If one require that the torsion vanish and that the covariant derivative be compatible with the metric then the $\Gamma^i{}_{jk}$ are given uniquely in terms of the $C^i{}_{jk}$.

For a general introduction to Kaluza-Klein theory and to references to the previous literature we refer to the review articles by Appelquist *et al.* [4], Bailin & Love [7] or Coquereaux & Jadczyk [24]. Model building using traditional Kaluza-Klein is developed, for example, by Kapetanakis & Zoupanos [65] and by Kubyshin *et al.* [70]. On the extended space-time one can consider gravity or one can consider, as a simpler problem, electromagnetism. This was first done some time ago by Forgács & Manton [46], Manton [83], Chapline & Manton [16], Fairlie [45] and Harnad *et al.* [54] The idea has a straightforward generalization to noncommutative Kaluza-Klein theory which we shall discuss in Section 4.1

3 Noncommutative Geometry

The aim of noncommutative geometry is to reformulate as much as possible the results of ordinary geometry in terms of an algebra of functions and to generalize them to the case of a general noncommutative (associative) algebra. The main notion which is lost when passing from the commutative to the noncommutative case is that of a point. The original noncommutative geometry is the quantized phase space of non-relativistic quantum mechanics. In fact Dirac in his historical papers in 1926 [32, 33] was aware of the possibility of describing phase-space physics in terms of the quantum analogue of the algebra of functions, which he called the quantum algebra, and using the quantum analogue of the classical derivations, which he called the quantum differentiations. And of course he was aware of the absence of localization, expressed by the Heisenberg uncertainty relation, as a central feature of these geometries. Inspired by work by von Neumann [94] for several decades physicists studied quantum mechanics and quantum field theory as well as classical and quantum statistical physics giving prime importance to the algebra of observables and considering the state vector as a secondary derived object. This work has much in common with noncommutative geometry. Only recently has an equivalent of an exterior derivative been introduced [18].

The motivation for introducing noncommutative geometry in Kaluza-Klein theory lies in the suggestion that space-time structure cannot be adequately described by ordinary geometry to all length scales, including those which are presumably relevant

when considering hidden dimensions. There is of course no reason to believe that the extra structure can be adequately described by the simple matrix geometries which we shall consider, although this seems well adapted to account for the finite particle multiplets observed in nature.

The simplest noncommutative algebras are the algebras M_n of $n \times n$ complex matrices. Let λ_a in M_n, for $1 \leq a \leq n^2 - 1$, be an antihermitian basis of the Lie algebra of the special unitary group SU_n. The product $\lambda_a \lambda_b$ can be written in the form

$$\lambda_a \lambda_b = \frac{1}{2} C^c{}_{ab} \lambda_c + \frac{1}{2} D^c{}_{ab} \lambda_c - \frac{1}{n} g_{ab}. \tag{37}$$

The g_{ab} are the components of the Killing metric; we shall use it to raise and lower indices. The $C^c{}_{ab}$ here are the structure constants of the group SU_n and $g_{cd} D^d{}_{ab}$ is trace-free and symmetric in all pairs of indices.

We introduce derivations e_a by

$$e_a f = [\lambda_a, f] \tag{38}$$

for an arbitrary matrix f. It is an elementary fact of algebra that any derivation X of M_n can be written as a linear combination $X = X^a e_a$ of the e_a with the X^a complex numbers. The complete set of all derivations of M_n will replace the space of all smooth vector fields on the hidden part of extended space-time.

We define the algebra of forms $\Omega^*(M_n)$ over M_n just as we did in the commutative case. First we define $\Omega^0(M_n)$ to be equal to M_n. Then we use (29) to define df. This means in particular that

$$d\lambda^a(e_b) = [\lambda_b, \lambda^a] = -C^a{}_{bc} \lambda^c. \tag{39}$$

The algebra of forms $\Omega^*(M_n)$ and the extension of the differential d is defined exactly as in Section 2. The big difference is that the algebra is not commutative and the algebra of forms is not graded commutative. Graded commutativity can be partially maintained however if instead of $d\lambda^a$ we use the 1-forms

$$\theta^a = \lambda_b \lambda^a d\lambda^b. \tag{40}$$

These 1-forms have a special relation with the derivations. Instead of (39) we have

$$\theta^a(e_b) = \delta^a_b \tag{41}$$

a fact which makes calculations easier since

$$df = e_a f \theta^a \tag{42}$$

as in (28). From (41) one can derive also the relations

$$\theta^a \theta^b = -\theta^b \theta^a, \qquad f\theta^b = \theta^b f \tag{43}$$

as well as

$$d\theta^a = -\frac{1}{2} C^a{}_{bc} \theta^b \theta^c. \tag{44}$$

From the generators θ^a we can construct a 1-form

$$\theta = -\lambda_a \theta^a. \tag{45}$$

Using θ we can rewrite (44) as

$$d\theta + \theta^2 = 0. \tag{46}$$

The interest in θ comes from the form

$$df = -[\theta, f] \tag{47}$$

for the differential of a matrix, an equation which follows directly from (42)

One can use matrix algebras to construct examples of differential calculi which have nothing to do with derivations. Consider the algebra M_n graded as in supersymmetry with even and odd elements and introduce a graded commutator between two matrices α and β as

$$[\alpha, \beta] = \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha \tag{48}$$

where $|\alpha|$ is equal to 0 or 1 depending on whether α is even or odd. One can define on M_n a graded derivation $\hat{d}$ by the formula

$$\hat{d}\alpha = -[\eta, \alpha], \tag{49}$$

where η is an arbitrary antihermitian odd element. Since η anti-commutes with itself we find that $\hat{d}\eta = -2\eta^2$ and for any α in M_n,

$$\hat{d}^2\alpha = [\eta^2, \alpha]. \tag{50}$$

The grading can be expressed as the direct sum $M_n = M_n^+ \oplus M_n^-$ of the even and odd elements of M_n. This decomposition is the analogue of (10). If n is even it is possible to impose the condition

$$\eta^2 = -1. \tag{51}$$

From (50) we see that $\hat{d}^2 = 0$ and the map (49) is a differential. In this case we shall write $\hat{d} = d$. We see that η must satisfy

$$d\eta + \eta^2 = 1, \tag{52}$$

an equation which is to be compared with (46). If we define for all $p \geq 0$

$$\Omega^{2p}(M_n^+) = M_n^+, \qquad \Omega^{2p+1}(M_n^+) = M_n^- \tag{53}$$

then we have defined a differential calculus over M_n^+. The differential algebra based on derivations can be imbedded in a larger algebra such that a graded extension of (47) exists for all elements [77]. In fact any differential calculus can be so extended.

As an example let $n = 2$. To within a normalization the matrices λ_a can be chosen to be the Pauli matrices. We define λ_1 and λ_2 to be odd and λ_3 and the identity even. The most general possible form for η is a linear combination of λ_1 and λ_2 and it can be normalized so that (51) is satisfied. Using $\Omega^*(M_2^+)$ one can construct a differential calculus over the algebra of functions on a double-sheeted space-time. This doubled-sheeted structure permits one [20] to introduce a description of parity breaking in the weak interactions.

If n is not even or, in general, if η^2 is not proportional to the unit element of M_n then $\hat{d}^2$ given by (50) will not vanish and M_n will not be a differential algebra. It is still possible however to construct over M_n^+ a differential calculus $\Omega^*(M_n^+)$ based on Formula (49). Essentially what one does is just eliminate the elements which are the image of $\hat{d}^2$ [21].

As an example let $n = 3$. There is a grading defined by the decomposition $3 = 2+1$ The most general possible form for η is

$$\eta = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ -a_1^* & -a_2^* & 0 \end{pmatrix}. \tag{54}$$

For no values of the a_i can (51) be satisfied. The general construction yields $\Omega^0(M_3^+) = M_3^+ = M_2 \times M_1$ and $\Omega^1(M_3^+) = M_3^-$ as in the previous example but after that the elimination of elements which are the image of $\hat{d}^2$ reduces the dimensions. One finds $\Omega^2(M_3^+) = M_1$ and $\Omega^p(M_3^+) = 0$ for $p \geq 3$.

Consider the ordinary Dirac operator $\slashed{D}$ and let ψ be a spinor and f a smooth function. It is straightforward to see that

$$e_a f i \gamma^a \psi = [\slashed{D}, f]\psi. \tag{55}$$

If we make the replacement $\gamma^a \mapsto \theta^a$ the left-hand side becomes equal to $idf\psi$. Because of the formal resemblance of (47) and (49) with this equation the matrices θ and η can be considered as generalized (antihermitian) Dirac operators. It is to be noticed that also $\slashed{D}^2 \neq 1$ and were one to use (55) to construct a differential one would have also to eliminate unwanted terms. The problem here is that $\theta^a \theta^b + \theta^b \theta^a = 0$ whereas $\gamma^a \gamma^b + \gamma^b \gamma^a \neq 0$. If we consider the algebra of functions $\mathcal{A}$ acting by multiplication on the Hilbert space $\mathcal{H}$ of spinors then the ordinary differential calculus can be described by the triple $(\mathcal{A}, \mathcal{H}, \slashed{D})$. As such it can be generalized to the noncommutative case [17, 19]. The triples for the examples above are $(M_2^+, \mathbb{C}^2, \eta)$ and $(M_3^+, \mathbb{C}^3, \eta)$ with η in M_2^- and M_3^- respectively.

One can study 'electromagnetism' on the algebras defined above, using the differential calculi. Consider first the algebra M_n with the differential calculus based on derivations. In the commutative case we neglected the Dirac structure and considered a 'spinor field' as an element of the algebra of functions. Here we do the same. We identify a 'spinor field' ψ as an element of the algebra M_n; it is a 'function' and it can be multiplied from the left by another arbitrary 'function' f. A covariant derivative is a map of the form (23) which for the same reasons we can write

$$D\psi = d\psi + \omega\psi. \tag{56}$$

Using the graded Leibniz rule (26) the covariant derivative can be extended to higher-order forms and the field strength Ω defined by the equation

$$D^2\psi = \Omega\psi. \tag{57}$$

By a simple calculation one finds that

$$\Omega = d\omega + \omega^2. \tag{58}$$

The extra term arises because the algebra is noncommutative. Again by strict analogy with the commutative case we define the gauge transformations to be the group U_n of unitary elements of M_n. It plays here the role of the *local* gauge transformations. The 1-form ω must take its values in the Lie algebra of U_n that is, the set of antihermitian elements of M_n. A particular choice of ω is $\omega = \theta$. It is easy to verify that θ is invariant under a gauge transformation. It makes sense then to decompose ω as a sum $\omega = \theta + \phi$ and one sees that ϕ transforms under the adjoint action of the group U_n: $\phi \mapsto g^{-1}\phi g$.

Expand $\phi = \phi_a \theta^a$. Then each ϕ_a is a matrix. Using the identities (44), (46) and (47) one sees that

$$\Omega = \frac{1}{2}\Omega_{ab}\theta^a\theta^b, \qquad \Omega_{ab} = [\phi_a\phi_b] - C^c{}_{ab}\phi_c. \tag{59}$$

One proceeds exactly in the same fashion with the algebra M_n^+ and the differential calculus based on the generalized Dirac operator η. One splits the gauge potential as a sum

$$\omega = \eta + \phi \tag{60}$$

and one finds, using the identity (49) with $\hat{d} = d$ and the identity (52), that

$$\Omega = 1 + \phi^2. \tag{61}$$

Recall that the right-hand side is a 2-form. In the two examples given above, with $n = 2$ and $n = 3$, it can be identified as a real number.

One can also study 'gravity' on M_n using the differential calculus based on derivations. One defines a metric by the condition that the θ^a be orthonormal with respect to the components of the Killing metric:

$$g(\theta^a \otimes \theta^b) = g^{ab}. \tag{62}$$

The unique metric-compatible torsion-free covariant derivative is given by

$$D\theta^a = -\frac{1}{2}C^a{}_{bc}\theta^b \otimes \theta^c. \tag{63}$$

On a matrix algebra there is a natural notion of integration defined by the trace. For this and other further developments we refer to the original literature. The basic texts on noncommutative geometry are the books by Connes [17, 19]. We refer also to a recent physically oriented book [77]. The idea of using derivations to define a differential calculus in the noncommutative case was first considered by Dubois-Violette [36]. The 1-forms θ^a were introduced and used to study noncommutative gauge theory in a series of articles by Dubois-Violette *et al.* [37, 38, 39, 40, 41] The differential calculus based on the generalized Dirac operator was introduced by Connes [17, 19]. It was applied to matrices by Connes & Lott [20, 21] and by Coquereaux *et al.* [25]. Other early references are the articles by Connes [18], Connes & Rieffel [22] and by Coquereaux [23]. It has been shown [82] that there is a sense in which the calculus based on M_2^+ and the operator η can be considered as a singular deformation of the calculus using M_2 and its derivations. The introduction of 'gravity' is much more difficult than 'electromagnetism' because of a technical problem coupled with the structure of the 1-forms. If one compares (23) with (32) one sees that whereas one must be able to multiply elements of $\mathcal{H}$ only from the left by elements of $\mathcal{A}$, one must be able to multiply elements of $\Omega^1(\mathcal{A})$ from the left and from the right. In the noncommutative case these two actions are not the same. A solution to this problem has been suggested by Mourad [84] and developed by Dubois-Violette *et al.* [42, 43] and others [81, 48].

4   Kaluza-Klein Theory Revisited

In traditional Kaluza-Klein theory the higher-order modes in the mode expansion of the field variables in the coordinates of the internal space are neglected, with the justification that they have all masses of the order of the Planck mass and would not be of interest in conventional physics. The alternative theory we here propose possesses

ab initio only a finite number of modes; there are no extraneous modes to truncate. We would like to suggest also that the noncommutative version of Kaluza-Klein theory is more natural than the traditional one in that a hand-waving argument can be given which allows one to think of the extra algebraic structure as being due to quantum fluctuations of the light-cone in ordinary 4-dimensional space-time. It has been argued that this structure remains as a 'classical shadow' of the fluctuations, of which the noncommutative structure of space-time itself is a higher-order correction. Let G be the gravitational constant and set $\bar{k} = G\hbar$. Let $\mathcal{A}_{\bar{k}}$ be the regularized algebra of observables of quantum field theory, including the regularizing gravitational field. If one lets $\bar{k} \to 0$ then the algebra $\mathcal{A}_{\bar{k}}$ becomes completely singular by assumption. It has no 'classical' limit. One can suppose however that some subalgebra $\mathcal{Z}_{\bar{k}} \subset \mathcal{A}_{\bar{k}}$ remains regular and has a commutative limit $\mathcal{Z}_0$ which one can identify as the algebra of functions on space-time. We have supposed further that some quasiclassical approximation exists which we can identify as a Kaluza-Klein extension of space-time [80, 67]. The origin of this argument is the old idea, due to Pauli and developed by Deser [29] and others [62], that perturbative ultraviolet divergences will one day be regularized by the gravitational field.

The version of Kaluza-Klein theory which we propose consists in replacing the k internal coordinates x^a by generators of a noncommutative algebra, for example the elements λ_a introduced in Section 3. This means that the k last components θ^a of the θ^i defined in Equation (19) must be replaced, for example by those defined by Equation (40). A 'moving frame' can be defined then by

$$\theta^i = (dx^\alpha, \lambda_b \lambda^a d\lambda^b). \tag{64}$$

We have considered here the internal structure formally as being of dimension $k = n^2 - 1$. This is however misleading since $n^2 - 1$ is the dimension of all the 'vector fields' on the algebraic structure, not the dimension of the tangent space at one point.

If the geometry is one of those based on the generalized Dirac operator η then the more abstract notation must be used since there is no basis θ^a and the total gauge potential ω must be written in the index-free notation. Using (60) one has

$$\omega = A + \eta + \phi \tag{65}$$

and one calculates the total curvature or field strength using the identity (52). Otherwise the development proceeds very much as in traditional electromagnetism. Equation (20) for the θ^a must be replaced by (44) since the internal 'space' is 'curved'. The integral over the internal space becomes a trace over the algebraic factor. As we have already mentioned there are two natural theories one can consider: the Maxwell-Dirac theory and the Einstein-Dirac theory.

4.1   Models with Electromagnetism

Most of the efforts to introduce noncommutative geometry into particle physics have been directed towards trying to find an appropriate noncommutative generalization of the idea mentioned at the end of Section 2. One studies electromagnetism on a noncommutative extension of space-time and one calculates how the particle and mass spectra vary as one varies the extra noncommutative algebra and the associated differential calculus. Much ingenuity has gone into these calculations which often involve very sophisticated mathematics but which ultimately reduce to simple manipulations with matrices.

The idea then is, for example, to consider the electromagnetic gauge potential $A = A_i \theta^i$ in an extended space-time but using the basis (64) instead of (19). Otherwise the formal manipulations are the same. One arrives at a unification of Yang-Mills and Higgs fields with the potential of the Higgs particle given by the curvature of the covariant derivative in the algebraic 'directions'. It is quartic in the field variables since the Yang-Mills action is quartic in the gauge potential. From the Expression (61) for the curvature, for example, one can see the origin of the Higgs potential normally introduced *ad hoc* to cause spontaneous symmetry breaking. Of course the differential calculus has in this case been chosen appropriately.

The simplest and most intuitive models are those based on derivations, introduced by Dubois-Violette *et al.* [37] and extended [38, 39, 40, 73, 8, 76] soon after. The models based on the generalized Dirac operator are less rigid and can be chosen to coincide with the Standard Model. The first example, constructed by Connes & Lott [20], was based on the differential calculus defined by Equation (53) for $n = 2$. The extension to $n = 3$ was given by Connes & Lott [21]. A different extension to $n = 4$ was developed concurrently by Coquereaux *et al.* [25, 28, 27], Scheck [89], Papadopoulos *et al.* [87]. Further developments were due to Iochum & Schücker [59], Papadopoulos & Plass [86] and Dimakis & Müller-Hoissen [30]. Several review articles have been written of these models. We refer, for example to the articles by Kastler [66], by Várilly & Gracia-Bondía [93] and by Kastler *et al.* [60]. A comparison of the two methods has been made by Dubois-Violette *et al.* [41] and others [82].

The weak interactions violate parity and this fact must be included in a realistic model. No derivation-based model with explicit parity violation has been developed; the models mentioned above rely implicitly on spontaneous parity-breaking mechanisms like the 'see-saw' mechanism. As we have already mentioned the double-sheeted structure of the Dirac-based models lends itself more readily to the introduction of explicit parity violation. We refer to Alvarez *et al.* [3] for a discussion of anomalies in this context.

Particle physics at the scale of grand unification has been examined from the point of view of noncommutative geometry by Chamseddine *et al.* [12, 13, 11], Batakis *et al.* [10, 9] and others [76, 85]. Supersymmetry has also been included [26, 55, 56]. In fact as was pointed out by Hussain & Thompson [58] the noncommutative models based on the differential calculi (53) are similar in structure to a 'supersymmetric' model proposed by Dondi & Jarvis some time ago [34]. Somewhat within the same context a completely different point of view of the role of noncommutative geometry has been given by Iochum *et al.* [61].

4.2   Models with Gravity

Very few of the results of the preceding subsection can be developed within the context of the Einstein-Dirac theory and none of them have as yet any significance for particle physics. We refer simply to the original literature. Gravity was first introduced in the context of noncommutative geometry by Dubois-Violette *et al.* [38] and developed in subsequent articles [72, 79, 80, 67, 52]. The definition of curvature remains a problem [43] as is the choice of action functional [18, 63, 52, 1]. A parallel development which treats gravity as an ordinary gauge field is due to Chamseddine *et al.* [14, 15] and others [90]. The details are given in the lecture by Chamseddine.

5   Noncommutative Space-Time

We saw in the Introduction that a field theory in a noncommutative version of space-time would have no ultraviolet divergences because there would be no points. We saw also that the ultimate use of noncommutative geometry as far as we are concerned is to describe quantum and/or classical gravity. In Section 4 we mentioned the old idea that ultraviolet divergences will one day be regularized by the gravitational field. The bridge between these ideas is the idea that noncommutative structure of space-time is due to the quantum fluctuations of the gravitational field [35, 80, 67]. The first mention of noncommuting coordinates in space-time in order to eliminate divergences was made by Snyder in 1947 [91, 92]. We refer also to the early article by Hellund & Tanaka [57] and to the more recent lecture notes by Bacry [6]. Although the position of a particle has no longer a well-defined meaning one can require that the Lorentz group act on the algebra. This was in fact the point which Snyder was the first to make and which distinguishes a noncommutative structure from the lattices which had been previously considered to represent the micro-texture of space-time. The space-time looks then like a solid which has a homogeneous distribution of dislocations but no disclinations. We can pursue this solid-state analogy and think of the ordinary Minkowski coordinates as macroscopic order parameters obtained by coarse-graining over scales less than the fundamental scale. They break down and must be replaced by elements of the algebra when one considers phenomena on these scales.

A simple model in two dimensions with euclidean signature has been introduced [74] and developed [75, 49, 77, 50, 51]. Although too simple to give much intuition about the 'correct' procedure it is an interesting example of the correlation between non-commutativity and curvature. A model in arbitrary dimension but with euclidean signature [78] is still in a preliminary state as is an example based on an extension of the quantum plane [31].

Quite generally one can address the question of how far it is possible to transcribe all of space-time physics into the language of noncommutative geometry. We have seen that a differential calculus can be constructed over an arbitrary associative algebra. This would permit the formulation of gauge theories in any geometry. In a less general setting a sort of Dirac operator has been proposed and a generalized integral [18]. A serious problem is that of quantization. The Standard Model is defined by a classical action which is assumed to contain implicitly all of high-energy physics. Quantum corrections are obtained by a standard quantization procedure. This quantization procedure has not been generalized to noncommutative models even in the simplest cases. The examples which have been used to propose classical actions which might be relevant in high-energy physics all involve simple matrix factors. They are quantized by first expanding the noncommutative fields in terms of ordinary space-time components and then quantizing the components. Under quantization the constraints on the model which come from the noncommutative geometry are lost [38, 2, 53].

To conclude we mention two other closely related approaches to the problem of the quantization of the gravitational field. It can be argued that since one has 'quantized' space one should also 'quantize' the Lorentz group. This idea leads to the notion of 'quantum spaces' and 'quantum groups'. They are described in some detail in the lectures by Castellani and by Wess. The theory of strings is based on the idea that the coordinates of (extended) space-time are fields on the world surfaces of string-like objects. When quantized they naturally become noncommuting objects. Under certain circumstances they even 'noncommute' classically [5].

Acknowledgments

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